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Galilei invariant theories: I. Constructions of indecomposable finite-dimensional representations of the homogeneous Galilei group: directly and via contractions

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Abstract
All indecomposable finite-dimensional representations of the homogeneous Galilei group which when restricted to the rotation subgroup are decomposed to spin-0, -1/2 and -1 representations are constructed and classified. These representations are also obtained via contractions of the corresponding representations of the Lorentz group. Finally, the obtained representations are used to derive a general Pauli anomalous interaction term as well as to deduce wave equations which describe Darwin and spin–orbit couplings of a Galilei particle interacting with an external electric field.

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1. Introduction

Unlike what most physicists might think of, the mathematical structure of the representations of the Galilei group is in many respects more complex and sophisticated than that of their relativistic counterparts. This is perhaps also the reason why unitary irreducible representations of the Poincaré group—the symmetry group of special theory of relativity—have been known nearly 20 years earlier than those of the Galilei group, even though the Galilei principle of relativity was discovered several centuries prior to the Einstein one.

The Galilei group and its representations are described in the Lévy-Leblond masterful survey [1] written almost 35 years ago. They form the group-theoretical basis for description of various physical predictions in non-relativistic classical mechanics and electrodynamics and in non-relativistic quantum mechanics as well (see also a more recent review [2]). These predictions are generally more than just some approximations to the relativistic results since only comparison of the predictions based on the Galilei group and its representations with those on the Poincaré group can indicate which predictions have the origin in the non-relativistic and which in the relativistic one. For instance, it was shown in [3, 4] and by Lévy-Leblond in [1, 5, 6] that the concept of spin of particles and of magnetic moments of particles have the origin already in the Galilean non-relativistic quantum mechanics and not as stated in many textbooks as a consequence of relativistic effects.

In the Galilei invariant framework, we shall consider first free particles and then particles interacting with external fields. The free particles can be described either by representations of the Galilei group (irreducible in the case of elementary particles and indecomposable for particles with internal structures) or, equivalently, by Galilei invariant wave equations. For interacting particles, the wave equations are more appropriate since they allow us to introduce interactions.

There exist three approaches how to formulate the Galilei invariant wave equations. The first is based on the fact that the Galilei group and the Galilei invariant equations can be obtained from the Poincaré group and Poincaré invariant equations, respectively, by a limiting procedure—the so-called Inönü–Wigner contraction [7].

The second approach consists of first writing the Poincaré invariant equations in the $(4 + 1)$-dimensional spacetime and then projecting them down to the $(3 + 1)$-dimensional Newtonian spacetime using the fact that the extended Galilei algebra in $(3 + 1)$ dimensions is a subalgebra of the Poincaré algebra in $(4 + 1)$ dimensions; for a connection of representations of these algebras see [8]. This approach, referred here as a projecting one, has been developed in [9–11] (see also [12]).

The third way to construct Galilei invariant theories consists in searching for these theories directly using the requirement of Galilei invariance and knowledge of representations of the Galilei group. We shall show in this paper that, in many respects, this latest approach is the most powerful and comprehensive. Moreover, it allows us to construct such consistent Galilei invariant equations of motion which are very difficult to derive using the contraction or projection methods. On the other hand, the direct search for wave equations invariant with respect to the inhomogeneous Galilei group needs a knowledge of the indecomposable finite-dimensional representations of its homogeneous Galilei subgroup, which has not been available till now.

Our paper serves the following four aims: (1) to describe all indecomposable finite-dimensional representations of the homogeneous Galilei group that are defined on spinor, scalar and vector representation spaces; (2) to specify all Galilei invariant bilinear forms, which facilitate derivation of various nonlinear Galilei wave equations; (3) to specify the reducible representations of the Lorentz group which lead to the found representations of the Galilei group via contractions; (4) to determine the Galilean spinor wave equations which include the Pauli anomalous terms.

In the next section, we define indecomposable finite-dimensional representations of the homogeneous Galilei group in general.

In sections 3 and 4, some of these representations are constructed explicitly, namely all those which when restricted to representations of the rotation subgroup of the homogeneous Galilei group are decomposed to spin-0, -1/2 and -1 representations. In addition, we present here also the complete list of bilinear forms invariant with respect to all found representations of the Galilei group. Section 5 contains various examples of the Galilean vectors. In section 6, we obtain the previously found representations of the homogeneous Galilei group via the Inönü–Wigner contraction of the corresponding representations of the Lorentz group. In section 7, the found representations are used to deduce the most general form of the Pauli interaction term which can be added to the Galilei invariant equation for spinors. Subsection 7.3 presents a simple Galilean system which describes the Darwin and spin–orbital interactions of particles with an external electric field. Finally, section 8 is devoted to discussions of the obtained results.

2. Definitions and properties of the Galilei group and its Lie algebra

The Galilei group $G(1,3)$ consists of the following transformations of time variable t and of space variables $\mathbf{x} = (x_1, x_2, x_3)$:

$$\begin{aligned} &t \to t' = t + a, \\ &\mathbf{x} \to \mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{v}t + \mathbf{b}, \end{aligned} \tag{1}$$

where a, $\mathbf{b}$ and $\mathbf{v}$ are real parameters of time translation, space translations and pure Galilei transformations, respectively, and the matrix $\mathbf{R}$ specifies rotations determined by the three parameters $\theta_1, \theta_2, \theta_3$.

The Galilei group includes a subgroup leaving invariant a point $\mathbf{x} = (0, 0, 0)$ at time $t = 0$. It is formed by all space rotations and pure Galilei transformations, i.e., by transformations (1) with $a = \mathbf{b} \equiv 0$. This subgroup is said to be *the homogeneous Galilei group* $HG(1,3)$. It is a semidirect product of the three-parameter commutative group of pure Galilei transformations with the rotation group. This group is not compact and does not have unitary finite-dimensional representations.

The Galilei group $G(1,3)$ is a semidirect product of its Abelian subgroup generated by time and space translations with the homogeneous Galilei group $HG(1,3)$.

Unitary representations of the Galilei group which are ordinary ones were described by Inönü and Wigner [3] in 1952, whereas those which are ray by Bargmann [4] in 1954. A nice review of these representations can be found in [1], see also [14].

However, a decisive role in the description of various finite-component Galilei invariant equations is played by finite-dimensional representations of the homogeneous Galilei group $HG(1,3)$. They were first studied according to our knowledge in paper [15]. Moreover, they have not been classified till now.

Let us recall that the representations of $HG(1,3)$ induce ray representations of the Galilei group $G(1,3)$ as well as ordinary representations of the extended Galilei group $G_m(1,3)$ which is a central extension of the Galilei group via a one-parameter subgroup. Both of them are realized in the space of (square integrable) n-component functions $\Psi(t, \mathbf{x})$ which for any transformation (1) cotransform in the following way [1]:

$$\Psi(\mathbf{x}, t) \to \Psi'(\mathbf{x}', t') = \mathrm{e}^{\mathrm{i}f(\mathbf{x},t)} T\Psi(\mathbf{x}, t), \tag{2}$$

where $\Psi(\mathbf{x}, t) = \mathrm{column}(\Psi_1(\mathbf{x}, t), \Psi_2(\mathbf{x}, t), \ldots, \Psi_n(\mathbf{x}, t))$ are n-component vectors from the representation space, T are $n \times n$ matrices depending on transformation parameters $\mathbf{v}$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$,

$$f(\mathbf{x}, t) = m\left(\mathbf{v}\cdot\mathbf{x} + \frac{v^2}{2}t + c\right)$$

is a phase which includes two parameters: m and c. For $m = 0$, the central extension is trivial and representations (2) are ordinary representations of $G(1,3)$. For m different from zero, the central extension is non-trivial and transformations (2) realize ray representations of the Galilei group $G(1,3)$ and ordinary representations of $G_m(1,3)$. Moreover, the transformation matrices T realize finite-dimensional representations of the homogeneous Galilei group $HG(1, 3)$.

Let us mention that realizations (2) are precisely those which are used in quantum mechanics and quantum field theory. These realizations are also essential to construct wave equations invariant w.r.t. the Galilei group [12].

Taking expressions (2) corresponding to the infinitesimal transformations (1) and treating c as an additional transformation parameter, we can calculate the 11-dimensional Lie algebra of the extended Galilei group. Basis elements of this algebra are of the following form:

$$\begin{aligned} &P_0 = \mathrm{i}\partial_0, \qquad P_a = -\mathrm{i}\partial_a, \qquad M = Im, \\ &J_a = -\mathrm{i}\varepsilon_{abc}x_b\partial_c + S_a, \\ &K_a = -\mathrm{i}x_0\partial_a - mx_a + \eta_a, \end{aligned} \tag{3}$$

where indices a, b and c run over the values $1, 2, 3$, I is the unit matrix, and S_a and η_a are matrices which satisfy the following commutation relations:

$$[S_a, S_b] = \mathrm{i}\varepsilon_{abc}S_c, \tag{4}$$

$$[\eta_a, S_b] = \mathrm{i}\varepsilon_{abc}\eta_c, \tag{5}$$

$$[\eta_a, \eta_b] = 0, \tag{6}$$

that is, they form a basis of the Lie algebra $hg(1, 3)$ of the homogeneous Galilei group $HG(1, 3)$. This algebra is isomorphic to the Lie algebra $e(3)$ of the Euclidean group.

Conditions (4)–(6) are necessary and sufficient in order generators (3) form a basis of a Lie algebra, namely the extended Galilei algebra satisfying the following relations:

$$\begin{aligned} &[J_a, J_b] = \mathrm{i}\epsilon_{abc}J_c, && [J_a, K_b] = \mathrm{i}\epsilon_{abc}K_c, \\ &[J_a, P_b] = \mathrm{i}\epsilon_{abc}P_c, && [K_a, P_0] = \mathrm{i}P_a, \\ &[K_a, P_b] = \mathrm{i}\delta_{ab}M, && [K_a, K_b] = 0, \\ &[P_a, P_b] = 0, && [P_0, P_a] = 0, \\ &[M, P_0] = 0, [M, P_a] = 0, && [M, J_a] = [M, K_a] = 0. \end{aligned} \tag{7}$$

Unfortunately, the problem of a complete classification of non-equivalent finite-dimensional indecomposable realizations of algebra (4)–(6) appears to be an unsolvable 'wild' algebraic problem. However, we shall show that this problem can be completely solved in the two important particular cases: for the purely spinor representations and vector representations. Notice that indecomposable representations of Lorentz and Poincaré groups were studied a long time ago, see paper [16] and references therein.

3. Spinor representations

The Lie algebra $hg(1, 3)$, defined by relations (4)–(6), includes the subalgebra $so(3)$ spanned on the basis elements S_1, S_2 and S_3. Without loss of generality, we suppose that representations of this subalgebra are Hermitian and completely reducible and shall search for representations of $hg(1, 3)$ in the $so(3)$ basis in which the Casimir operator of $so(3)$ is diagonal.

Irreducible representations of $so(3)$ are labelled by integers or half-integers s. Let $\tilde{s}$ be the highest value of s which appears in the decomposition of a reducible representation of $so(3)$ subduced by the indecomposable representation of $hg(1, 3)$. We shall call the related carrier space of this representation of $hg(1, 3)$ *the representation space of spin* $\tilde{s}$.

Note that it is reasonable to search for representations with integer and half-integer $\tilde{s}$ separately since neither rotations generated by spin matrices S_a nor Galilean boosts generated by matrices η_a can mix states with integer and half-integer spins. It is the general property of matrices satisfying relations (4), (5) [13].

Consider finite-dimensional indecomposable representations of algebra $hg(1,3)$ of spin $\tilde{s} = \frac{1}{2}$. Then the corresponding matrices S_1, S_2 and S_3 can be decomposed to a direct sum of the irreducible representations $D(1/2)$ of algebra $so(3)$:

$$S_a = \tfrac{1}{2} I_{n\times n} \otimes \sigma_a, \tag{8}$$

where $I_{n\times n}$ is the $n \times n$ unit matrix with a finite n and σ_a are the usual Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From (8) and (5), the generic form of the related matrices η_a is

$$\eta_a = A_{n\times n} \otimes \sigma_a, \tag{9}$$

where $A_{n\times n}$ is an $n \times n$ matrix. The commutativity of matrices (9) leads to the nilpotency condition

$$A_{n\times n}^2 = 0. \tag{10}$$

Thus, without any loss of generality, $A_{n\times n}$ may be expressed as a direct sum of 2×2 Jordan cells and zero matrix.

Because of (10) there exist only two different indecomposable representations of algebra $hg(1,3)$ defined on spin-1/2 carrier space:

$$S_a = \tfrac{1}{2}\sigma_a, \qquad \eta_a = 0$$

and

$$S_a = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix}, \qquad \eta_a = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ \sigma_a & 0 \end{pmatrix}. \tag{11}$$

The corresponding vectors from the representation space are two-component spinors $\varphi(\mathbf{x}, t)$, as well as four-component bispinors $\Psi = \binom{\varphi_1(\mathbf{x},t)}{\varphi_2(\mathbf{x},t)}$ with two-component φ_1 and φ_2, respectively. When t and $\mathbf{x}$ undergo a Galilean transformation (1), then φ cotransforms as

$$\varphi(\mathbf{x}, t) \to \varphi'(\mathbf{x}', t') = \mathrm{e}^{\mathrm{i}m(\mathbf{v}\cdot\mathbf{x}+\mathbf{v}^2 t/2+c)} \left(\cos\frac{\theta}{2} + \mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\theta}}{\theta}\sin\frac{\theta}{2}\right)\varphi(\mathbf{x}, t)$$

while the transformation law for components of Galilean bispinor is

$$\varphi_1(\mathbf{x}, t) \to \varphi_1'(\mathbf{x}', t') = \mathrm{e}^{\mathrm{i}m(\mathbf{v}\cdot\mathbf{x}+\mathbf{v}^2 t/2+c)} \left(\cos\frac{\theta}{2} + \mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\theta}}{\theta}\sin\frac{\theta}{2}\right)\varphi_1(\mathbf{x}, t),$$

$$\begin{aligned}\varphi_2(\mathbf{x}, t) \to \varphi_2'(\mathbf{x}', t') = \mathrm{e}^{\mathrm{i}m(\mathbf{v}\cdot\mathbf{x}+\mathbf{v}^2 t/2+c)} &\left(\left(\cos\frac{\theta}{2} + \mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\theta}}{\theta}\sin\frac{\theta}{2}\right)\varphi_2(\mathbf{x}, t)\right. \\ &\left. + \frac{1}{2}\left(\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{v}\cos\frac{\theta}{2} - (\boldsymbol{\theta}\cdot\mathbf{v} + \mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{\theta}\times\mathbf{v})\frac{\sin\frac{\theta}{2}}{\theta}\right)\varphi_1(\mathbf{x}, t)\right).\end{aligned}$$

We use the notation $\theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}$. Invariants of these transformations which are independent of t and $\mathbf{x}$ are arbitrary functions of $\varphi^\dagger\varphi$, $\varphi_1^\dagger\varphi_1$ and $\varphi_1^\dagger\varphi_2 + \varphi_2^\dagger\varphi_1$.

4. Vector representations

In this section, we examine the finite-dimensional indecomposable representations of the algebra $hg(1,3)$ defined on vector, or spin-1, representation spaces. The corresponding matrices S_a can be expressed as direct sums of spin-1 and spin-0 matrices:

$$S_a = \begin{pmatrix} I_{n\times n} \otimes s_a & \cdot \\ \cdot & \mathbf{0}_{m\times m} \end{pmatrix}. \tag{12}$$

The symbols $I_{n\times n}$ and $\mathbf{0}_{m\times m}$ denote the $n \times n$ unit matrix and $m \times m$ zero matrix, respectively, $s_a(a = 1, 2, 3)$ are 3×3 matrices of spin equal to 1 for which we choose the following realization:

$$s_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad s_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

The general form of matrices η_a which satisfy relations (5) with matrices (12) is given by the following formulae (see e.g. [13]):

$$\eta_a = \begin{pmatrix} A \otimes s_a & B \otimes k_a^\dagger \\ C \otimes k_a & \mathbf{0}_{m\times m} \end{pmatrix}. \tag{14}$$

A, B and C are matrices of dimensions $n \times n$, $n \times m$ and $m \times n$, respectively, and k_a are 1×3 matrices of the form

$$k_1 = (\mathrm{i}, 0, 0), \qquad k_2 = (0, \mathrm{i}, 0), \qquad k_3 = (0, 0, \mathrm{i}). \tag{15}$$

The matrices (12) and (14) satisfy conditions (5) with any A, B and C. Substituting (14) into (6) and using the relations

$$s_a k_b^\dagger = \mathrm{i}\varepsilon_{abc} k_c^\dagger, \qquad k_a s_b = \mathrm{i}\varepsilon_{abc} k_c,$$
$$[s_a, s_b] = k_a^\dagger k_b - k_b^\dagger k_a = \mathrm{i}\varepsilon_{abc} s_c,$$
$$k_a k_b^\dagger - k_b k_a^\dagger = 0,$$

we obtain the following equations for matrices A, B and C:

$$A^2 + BC = 0, \tag{16}$$
$$CA = 0, \qquad AB = 0. \tag{17}$$

We note that relations (16) are invariant w.r.t. the following transformations:

$$A \to A' = \alpha W A W^{-1}, \qquad B \to B' = \alpha W B V^{-1}, \qquad C \to C' = \alpha V C W^{-1}, \tag{18}$$

where α is a complex non-zero multiplier, and W and V are invertible matrices of dimensions $n \times n$ and $m \times m$, respectively. Sets of matrices $\{A, B, C\}$ and $\{A', B', C'\}$ connected by relations (18) will be treated as equivalent.

The solution of the matrix problem defined by equations (16) and (17) is relatively easy to handle; the detailed calculations are presented in the appendix. Namely, there exist ten non-equivalent indecomposable sets of matrices $\{A, B, C\}$, which can be labelled by triplets of numbers n, m, λ where n and m take the values

$$-1 \leqslant (n - m) \leqslant 2, \qquad n \leqslant 3 \tag{19}$$

and define dimensions of these matrices as in equation (14), $\lambda = \mathrm{Rank}B$, whose values are

$$\lambda = \begin{cases} 0, & \text{if } m = 0, \\ 1, & \text{if } m = 2 \text{ or } n - m = 2,\ m \neq 0, \\ 0, 1, & \text{if } m = 1,\ n \neq 3. \end{cases} \tag{20}$$

The list of non-equivalent sets of indecomposable matrices $\{A, B, C\}$ is given in lemma 2 and the related matrices S_a and η_a are given in table 1.

Any set of matrices S_a, η_a given in table 1 yields a finite-dimensional indecomposable representation of algebra $hg(1, 3)$ which generates a representation of the extended Galilei algebra via (3). From our analysis, there exist ten indecomposable vector representations of $hg(1, 3)$.

The finite transformations corresponding to these realizations are found by integrating the Lie equations for generators given in (3) and table 1. For a specific representation $D(n, m, \lambda)$, they have the following forms.

Table 1. Vector representations: spin matrices S_a and boost matrices η_a where s_a and k_a are matrices (13).

Representation $D(n, m, \lambda)$	S_a	η_a
$D(0, 0, 0)$	0	0
$D(1, 0, 0)$	s_a	$\mathbf{0}_{3\times3}$
$D(1, 1, 0)$	$\begin{pmatrix} s_a & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ k_a & 0 \end{pmatrix}$
$D(1, 1, 1)$	$\begin{pmatrix} s_a & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{0}_{3\times3} & k_a^\dagger \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix}$
$D(1, 2, 1)$	$\begin{pmatrix} s_a & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{3\times1} & 0 & 0 \\ \mathbf{0}_{3\times1} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{0}_{3\times3} & k_a^\dagger & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 0 & 0 \\ k_a & 0 & 0 \end{pmatrix}$
$D(2, 0, 0)$	$\begin{pmatrix} s_a & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & s_a \end{pmatrix}$	$\begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ s_a & \mathbf{0}_{3\times3} \end{pmatrix}$
$D(2, 1, 0)$	$\begin{pmatrix} s_a & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & s_a & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ s_a & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ k_a & \mathbf{0}_{1\times3} & 0 \end{pmatrix}$
$D(2, 1, 1)$	$\begin{pmatrix} s_a & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & s_a & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{0}_{3\times3} & s_a & k_a^\dagger \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix}$
$D(2, 2, 1)$	$\begin{pmatrix} s_a & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & s_a & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 & 0 \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ s_a & \mathbf{0}_{3\times3} & k_a^\dagger & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 & 0 \\ k_a & \mathbf{0}_{1\times3} & 0 & 0 \end{pmatrix}$
$D(3, 1, 1)$	$\begin{pmatrix} s_a & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & s_a & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & s_a & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ s_a & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & s_a & \mathbf{0}_{3\times3} & k_a^\dagger \\ -k_a & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix}$

- $D(0, 0, 0)$: the related representation space is formed by a field S invariant under rotations and which transforms under a Galilean boost as

$$S \to \mathrm{e}^{imf} S, \tag{21}$$

where $f = \mathbf{v} \cdot \mathbf{x} + \mathbf{v}^2 t/2$. Such transformations keep invariant the bilinear form $I_1 = S^* S$.
- $D(1, 0, 0)$: the Galilean 3-vectors $\mathbf{R} = \text{column}(R_1, R_2, R_3)$ transform as vectors under rotations:

$$\mathbf{R} \to \mathbf{R} \cos\theta + \frac{\boldsymbol{\theta} \times \mathbf{R}}{R} \sin\theta + \frac{\boldsymbol{\theta}(\boldsymbol{\theta} \cdot \mathbf{R})}{\theta^2} (1 - \cos\theta),$$

and via

$$\mathbf{R} \to \mathrm{e}^{imf} \mathbf{R}, \tag{22}$$

under the Galilean boosts. Then, the bilinear form $I_2 = \mathbf{R}^* \cdot \mathbf{R}$ is invariant under Galilean transformations.
- $D(1, 1, 0)$: Galilean 4-vector $\Psi_4 = (\mathbf{U}, S)$, where $\mathbf{U}$ is a vector and S is a scalar under the rotation transformations. The action of Galilean boosts on $\mathbf{U}$ and S can be written as

$$\begin{aligned} &\mathbf{U} \to \mathrm{e}^{imf} (\mathbf{U} + \mathbf{v} S), \\ &S \to \mathrm{e}^{imf} S. \end{aligned} \tag{23}$$

The corresponding invariant of Galilei group is of the form $I_1 = S^* S$.

- $D(1, 1, 1)$: the second Galilean 4-vector $\tilde{\Psi}_4 = (\mathbf{R}, P)$, where $\mathbf{R}$ and P transform under the Galilean boost as

$$\begin{aligned} \mathbf{R} &\to \mathrm{e}^{imf}\mathbf{R}, \\ P &\to \mathrm{e}^{imf}(P + \mathbf{v}\cdot\mathbf{R}). \end{aligned} \tag{24}$$

  The invariant form for these (and rotation) transformations can be written as $I_2 = \mathbf{R}^*\cdot\mathbf{R}$.
- $D(1, 2, 1)$: the Galilean 5-vector $\Psi_5 = (S, \mathbf{U}, Q)$, where S and Q are scalars and $\mathbf{U}$ is a vector with respect to rotations. Under a Galilean boost, they transform as

$$\begin{aligned} S &\to \mathrm{e}^{imf}S, \\ \mathbf{U} &\to \mathrm{e}^{imf}(\mathbf{U} + \mathbf{v}S), \\ Q &\to \mathrm{e}^{imf}\left(Q + \mathbf{v}\cdot\mathbf{U} + \tfrac{1}{2}\mathbf{v}^2 S\right), \end{aligned} \tag{25}$$

  and the invariants of these transformations are $I_2 = S^*S$ and $I_3 = S^*Q + SQ^* - \mathbf{U}^*\mathbf{U}$.
- $D(2, 0, 0)$: the Galilean 6-vectors (bi-vectors) $\Psi_6 = (\mathbf{R}, \mathbf{W})$, which under the Galilean boosts transform as

$$\begin{aligned} \mathbf{R} &\to \mathrm{e}^{imf}\mathbf{R}, \\ \mathbf{W} &\to \mathrm{e}^{imf}(\mathbf{W} + \mathbf{v}\times\mathbf{R}), \end{aligned} \tag{26}$$

  and the corresponding invariants are $I_2 = \mathbf{R}^*\cdot\mathbf{R}$ and $I_4 = \mathbf{R}^*\cdot\mathbf{W} + \mathbf{R}\cdot\mathbf{W}^*$.
- $D(2, 1, 0)$: the Galilean 7-vectors $\Psi_7 = (\mathbf{R}, \mathbf{W}, P)$. Under the Galilean boost, its components transform according to

$$\begin{aligned} \mathbf{R} &\to \mathrm{e}^{imf}\mathbf{R}, \\ P &\to \mathrm{e}^{imf}(P + \mathbf{v}\cdot\mathbf{R}), \\ \mathbf{W} &\to \mathrm{e}^{imf}(\mathbf{W} + \mathbf{v}\times\mathbf{R}). \end{aligned} \tag{27}$$

  The corresponding invariants are the same as the previous case: $I_2 = \mathbf{R}^*\cdot\mathbf{R}$ and $I_4 = \mathbf{R}^*\cdot\mathbf{W} + \mathbf{R}\cdot\mathbf{W}^*$.
- $D(2, 1, 1)$: the second 7-vector $\tilde{\Psi}_7 = (\mathbf{K}, \mathbf{R}, S)$ with $\mathbf{K}$ being a 3-vector which under the Galilean boost transforms as

$$\mathbf{K} \to \mathrm{e}^{imf}\left(\mathbf{K} + \mathbf{v}\times\mathbf{R} + \mathbf{v}S\right), \tag{28}$$

  and with $\mathbf{R}$, $\mathbf{S}$ which transform as

$$S \to \mathrm{e}^{imf}S, \mathbf{R} \to \mathrm{e}^{imf}\mathbf{R}. \tag{29}$$

  The corresponding invariants are $I_1 = S^*S$ and $I_2 = \mathbf{R}^*\cdot\mathbf{R}$.
- $D(2, 2, 1)$: Galilean 8-vector $\Psi_8 = (\mathbf{R}, \mathbf{K}, P, S)$, whose components with the Galilean boost transform as

$$\begin{aligned} S &\to \mathrm{e}^{imf}S, \mathbf{R} \to \mathrm{e}^{imf}\mathbf{R}, \\ \mathbf{K} &\to \mathrm{e}^{imf}\left(\mathbf{K} + \mathbf{v}\times\mathbf{R} + \mathbf{v}S\right), \\ P &\to \mathrm{e}^{imf}(P + \mathbf{v}\cdot\mathbf{R}). \end{aligned} \tag{30}$$

  The invariants of these transformations are $I_1 = S^*S$, $I_2 = \mathbf{R}^*\cdot\mathbf{R}$ and $I_5 = S^*P + SP^* - \mathbf{K}\cdot\mathbf{R}^* - \mathbf{K}^*\cdot\mathbf{R}$.
- $D(3, 1, 1)$: the 10-vector fields $\Psi_{10} = (\mathbf{R}, \mathbf{W}, \mathbf{N}, P)$ combine three 3-vectors $\mathbf{R}, \mathbf{W}, \mathbf{N}$ and one scalar P. They cotransform under the boost transformations via

$$\begin{aligned} \mathbf{R} &\to \mathrm{e}^{imf}\mathbf{R}, \\ \mathbf{W} &\to \mathrm{e}^{imf}(\mathbf{W} + \mathbf{v}\times\mathbf{R}), \\ \mathbf{N} &\to \mathrm{e}^{imf}\left(\mathbf{N} + \mathbf{v}\times\mathbf{W} + \mathbf{v}P + \mathbf{v}(\mathbf{v}\cdot\mathbf{R}) - \tfrac{1}{2}\mathbf{v}^2\mathbf{R}\right), \\ P &\to \mathrm{e}^{imf}(P + \mathbf{v}\cdot\mathbf{R}). \end{aligned} \tag{31}$$

  The invariants of these transformations are $I_2 = \mathbf{R}^*\cdot\mathbf{R}$, $I_4 = \mathbf{R}^*\cdot\mathbf{W} + \mathbf{R}\cdot\mathbf{W}^*$ and $I_6 = P^*P + \mathbf{W}\cdot\mathbf{W}^* - \mathbf{R}\cdot\mathbf{N}^* - \mathbf{N}\cdot\mathbf{R}^*$. In section 7, we shall utilize this vector in the Galilean electromagnetism.

Thus, in addition to Galilean scalar S, there exist nine Galilean vectors enumerated in the above items. We see that the number of such vectors is notably larger than in the case of proper Lorentz group, where there are only three irreducible multiplets whose components transform as vectors or scalars under rotations, namely a 4-vector and self-dual and anti-self-dual components of an antisymmetric tensor [17].

5. Examples of Galilean vectors

In the previous section, we have described finite-dimensional indecomposable Galilean vectors and presented explicitly their group transformations and invariants. They form the main tool for constructing various Galilei invariant models. In particular, by using the realizations of matrices S_a and η_a given in table 1, it is possible to describe all non-equivalent Galilei invariant first-order partial differential equations for vector fields. Here we present some important examples of Galilean vectors.

Example 1. Generators P_0 and P_a of time and space translations defined by relations (7) and mass m form a Galilean 5-vector of type Ψ_5 provided we identify $P_0 \sim Q, \mathbf{P} \sim \mathbf{U}, m \sim S$. For $m = 0$, this 5-vector is reduced to a 4-vector of type $\tilde{\Psi}_4$.

Example 2. 5-potential of the Galilean electromagnetic field [18] $\hat{A} = (A_0, \mathbf{A}, A_4)$ with the transformation law

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \mathbf{v}A_4, \\ A_0' &= A_0 + \mathbf{v}\cdot\mathbf{A} + \tfrac{1}{2}\mathbf{v}^2 A_4, \\ A_4' &= A_4 \end{aligned} \tag{32}$$

is an example of Galilean 5-vector field with zero mass, which is a carrier space of the representation $D(1, 2, 1)$ described in table 1.

Example 3. The field strength tensor of the Galilean electromagnetic field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{33}$$

where $\mu, \nu = 0, 1, 2, 3, 4$ and by definition $\partial_4 A_\mu = 0$, is the example of massless 10-vector which transforms in accordance with (31). The explicit relation between components of $F_{\mu\nu}$ and Ψ_{10} is given by the following formula:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -N_1 & -N_2 & -N_3 & P \\ N_1 & 0 & W_3 & -W_2 & R_1 \\ N_2 & -W_3 & 0 & W_1 & R_2 \\ N_3 & W_2 & -W_1 & 0 & R_3 \\ -P & -R_1 & -R_2 & -R_3 & 0 \end{pmatrix}, \tag{34}$$

where

$$\begin{aligned} P &= \partial_0 A_4, \\ \mathbf{W} &= \nabla \times \mathbf{A}, \\ \mathbf{N} &= \nabla A_0 - \partial_0 \mathbf{A}, \\ \mathbf{R} &= \nabla A_4. \end{aligned} \tag{35}$$

Some subsets of components of $F_{\mu\nu}$ form carrier spaces for other representations of the Galilei group. For example, if $R_a = F_{4a} \equiv 0$ then the remaining components of $F_{\mu\nu}$ transform as a 7-vector of type Ψ_7. The complete list of various vectors which can be formed using components of $F_{\mu\nu}$ is presented in section 7.

Example 4. As another example of a Galilean field can be considered the matrix 5-vector $\Psi_5 = (S, U_1, U_2, U_3, Q)$ with components

$$\begin{aligned} S &= \beta_0 = \tfrac{1}{\sqrt{2}}(\gamma_0 + \gamma_4), \\ Q &= \beta_4 = \tfrac{1}{\sqrt{2}}(\gamma_0 - \gamma_4), \\ U_a &= \beta_a = \gamma_a, \end{aligned} \tag{36}$$

where γ_μ are Dirac matrices, $\mu = 0, \ldots, 4$, $a = 1, 2, 3$. Such a set of matrices commutes as a 5-vector with the Galilei generators (3) if we choose

$$S_a = \tfrac{1}{4}\varepsilon_{abc}\gamma_b\gamma_c, \qquad \eta_a = \tfrac{1}{2\sqrt{2}}(\gamma_0 + \gamma_4)\gamma_a. \tag{37}$$

Example 5. As in the previous two examples, the matrix tensor

$$S_{\mu\nu} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu \tag{38}$$

transforms like $F_{\mu\nu}$, that is, as a 10-vector. Moreover, it is possible to form the following matrix vectors:

$$\begin{aligned} &\tilde{\Psi}_4 : (R_a = S_{0a}, P = S_{40}), \\ &\tilde{\Psi}_6 : \left(W_a = \tfrac{1}{2}\varepsilon_{abc}S_{bc}, R_a = S_{0a}\right), \\ &\tilde{\Psi}_7 : \left(R_a = S_{0a}, W_a = \tfrac{1}{2}\varepsilon_{abc}S_{bc}, P = S_{40}\right). \end{aligned} \tag{39}$$

6. Contractions of representations of Lorentz algebra

It is well known that the Galilei group (algebra) and (some of) its representations can be obtained from the Poincaré group (algebra) and its appropriate representations by a limiting procedure called 'contraction'. The process of contraction has, by now, an extensive literature. First, it was proposed for Lie algebras by Segal [19] and in more specific forms by Inönü and Wigner [7], by Saletan [20], by Doebner and Melsheimer [21] and many others; see the excellent review article of Lôhmus [22] and references cited therein. A more recent review can be found in [23].

The Lie algebra of a given Lie group is defined via commutation relations for basis elements. As shown by Cartan, whenever we change the basis by a non-singular transformation we come to algebra isomorphic to the original one. However, if the transformation is singular, a new algebra may be received, provided this transformation leads to well-defined commutation relations for the transformed basis elements.

In the simplest case, a *contraction* is a limit procedure which transforms an N-dimensional Lie algebra $\mathcal{L}$ into an non-isomorphic Lie algebra $\mathcal{L}'$, also with N dimensions. The commutation relations of a *contracted Lie algebra* $\mathcal{L}'$ are given by

$$[x, y]' \equiv \lim_{\varepsilon\to\varepsilon_0} W_\varepsilon^{-1}([W_\varepsilon(x), W_\varepsilon(y)], \tag{40}$$

where $W_\varepsilon \in GL(N, k)$ is a non-singular linear transformation of $\mathcal{L}'$, with ε_0 being a singularity point of its inverse W_ε^{-1}.

The papers [19–23] indicate different ways of performing this (and more general) singular transformation and necessary and sufficient conditions that a given Lie algebra can be contracted into another one. However, there is no regular way to obtain representations of the contracted algebra starting with the representations of the initial one. Namely, in contracting representations we meet the following difficulties of principle:

(i) Contracting the faithful representation of a given Lie algebra, we obtain in general a non-faithful representation of the resulting Lie algebra since a part of generators is represented trivially.

(ii) The resulting (contracted) algebra is always non-compact. Hence, any contraction of a Hermitian irreducible representation of some compact Lie algebra, which is always *finite* dimensional, has to yield at the end an *infinite*-dimensional Hermitian irreducible representation of the non-compact Lie algebra.

Inönü and Wigner in [7] mentioned possible ways of treating the difficulties, one of them will be used in the following.

There is a simple contraction procedure (the Inönü–Wigner contraction) connecting the Lie algebra $so(1,3)$ of the Lorentz group with algebra $hg(1,3)$. The related transformation W does not change basis elements of $so(1,3)$ forming its subalgebra $so(3)$ while the remaining basis elements are multiplied by a small parameter ε which tends to zero [7].

Contractions of Lie groups and their representations with the coordinate free method were studied in [24] where contraction of representations of the de Sitter group was carried out.

Here we find representations of the Lorentz group whose contraction makes it possible to obtain found realizations of the homogeneous Galilei group. A specific feature of these representations is that they are in a general case completely reducible while the corresponding contracted representations of the Galilei algebra are indecomposable ones.

Let $S_{\mu\nu}, \mu, \nu = 0, 1, 2, 3$ are matrices realizing a representation of the Lorentz algebra, i.e., satisfying the relations

$$[S_{\mu\nu}, S_{\lambda\sigma}] = \mathrm{i}(g_{\mu\lambda}S_{\nu\sigma} + g_{\nu\sigma}S_{\mu\lambda} - g_{\nu\lambda}S_{\mu\sigma} - g_{\mu\sigma}S_{\nu\lambda}) \tag{41}$$

with $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The contraction procedure consists in transition to a new basis

$$S_{ab} \to S_{ab}, \qquad S_{0a} \to \varepsilon S_{0a}$$

and simultaneous transformation of all basis elements $S_{\mu\nu} \to S'_{\mu\nu} = U S_{\mu\nu} U^{-1}$ with a matrix U depending on ε. Moreover, u should depend on ε in a tricky way, such that all the transformed generators $S'_{\mu\nu}$ are kept non-trivial when $\varepsilon \to 0$ [7].

We suppose that representations obtained by the contraction are indecomposable representations $D(n, m, \lambda)$ described in section 4. To construct representations of Lorentz algebra which can be contracted to representations $D(n, m, \lambda)$, we use the following observation.

Lemma 1. *Let $\{S_a, \eta_a\}$ be an indecomposable set of matrices realizing one of the representations $D(1,1,0), D(1,1,1), D(1,2,1), D(2,0,0)$ or $D(3,1,1)$ presented in table 1. Then matrices*

$$S_{ab} = \varepsilon_{abs}S_c, \qquad S_{0a} = \nu\big(\eta_a - \eta_a^\dagger\big), \tag{42}$$

where $\nu = 1$ for representations $D(1,1,0), D(1,1,1) D(2,0,0)$ and $\nu = \frac{1}{\sqrt{2}}$ for representations $D(1,2,1), D(3,1,1)$, form a basis of the Lie algebra of the Lorentz group.

The proof is reduced to the direct verification that for all basis elements S_a, η_a of the homogeneous Galilei algebra the corresponding linear combinations (42) satisfy relations (41), i.e., do form a basis of the Lorentz algebra.

For representations $D(1,0,0), (D(2,1,1), D(2,1,0))$ and $D(2,2,1)$, the related matrices (42) do not form a basis of the Lie algebra. Nevertheless, it is possible to find

Table 2. Representations of the Lorentz algebra and contracting matrices.

Representations of algebra $hg(1,3)$	Representations of algebra $so(1,3)$	Basis elements $S_{\mu\nu}$	Contracting matrix U
$D(1,0,0)$	$D(1,0)$	(43)	$I_{3\times3}$
$D(1,1,0)$	$D\left(\frac{1}{2},\frac{1}{2}\right)$	(42)	$\begin{pmatrix} I_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \varepsilon^{-1} \end{pmatrix}$
$D(1,1,1)$	$D\left(\frac{1}{2},\frac{1}{2}\right)$	(42)	$\begin{pmatrix} I_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \varepsilon \end{pmatrix}$
$D(1,2,1)$	$D\left(\frac{1}{2},\frac{1}{2}\right)\oplus D(0,0)$	(42)	$\begin{pmatrix} I_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \varepsilon & 0 \\ \mathbf{0}_{1\times3} & 0 & \varepsilon^{-1} \end{pmatrix}$
$D(2,0,0)$	$D(0,1)\oplus D(1,0)$	(42)	$\begin{pmatrix} I_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \varepsilon^{-1}I_{3\times3} \end{pmatrix}$
$D(2,1,0)$	$D\left(\frac{1}{2},\frac{1}{2}\right)\oplus D(1,0)$	(44)	$\begin{pmatrix} I_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \varepsilon^{-1}I_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \varepsilon^{-1} \end{pmatrix}$
$D(2,1,1)$	$D\left(\frac{1}{2},\frac{1}{2}\right)\oplus D(1,0)$	(44)	$\begin{pmatrix} I_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \varepsilon I_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \varepsilon \end{pmatrix}$
$D(2,2,1)$	$D\left(\frac{1}{2},\frac{1}{2}\right)\oplus$ $D(0,0)\oplus D(0,0)$	(45)	$\begin{pmatrix} I_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \varepsilon^{-1}I_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 1 & 0 \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 & \varepsilon^{-1} \end{pmatrix}$
$D(3,1,1)$	$D(0,1)\oplus D(1,0)$ $\oplus D\left(\frac{1}{2},\frac{1}{2}\right)$	(42)	$\begin{pmatrix} I_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \varepsilon^{-1}I_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \varepsilon^{-2}I_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \varepsilon^{-1} \end{pmatrix}$

the corresponding generators of the Lorentz algebra starting with its known representations of dimensions 3×3, 7×7 and 8×8, respectively. We choose these representations in the following forms:

$$S_{ab} = \varepsilon_{abc}s_c, \quad S_{0a} = \mathrm{i}s_a, \tag{43}$$

$$S_{ab} = \varepsilon_{abc}\begin{pmatrix} s_c & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & s_c & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix}, \qquad S_{0a} = \frac{1}{2}\begin{pmatrix} \mathrm{i}s_a & -s_a & \mathrm{i}\sqrt{2}k_a^\dagger \\ s_a & \mathrm{i}s_a & -\sqrt{2}k_a^\dagger \\ \mathrm{i}\sqrt{2}k_a & \sqrt{2}k_a & 0 \end{pmatrix} \tag{44}$$

and

$$S_{ab} = \varepsilon_{abc}\begin{pmatrix} s_c & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & s_c & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 & 0 \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 & 0 \end{pmatrix}, \qquad S_{0a} = \frac{1}{2}\begin{pmatrix} \mathrm{i}s_a & -s_a & \mathrm{i}k_a^\dagger & k_a^\dagger \\ s_a & \mathrm{i}s_a & -k_a^\dagger & \mathrm{i}k_a^\dagger \\ \mathrm{i}k_a & k_a & 0 & 0 \\ -k_a & \mathrm{i}k_a & 0 & 0 \end{pmatrix}, \tag{45}$$

where s_a and k_a are matrices given by equations (13) and (15).

Matrices (42)–(45) form bases of representations of the Lorentz algebra which are in general reducible. These representations (together with the related realizations of the homogeneous Galilei algebra given in table 1) are enumerated in table 2.

Thus, we obtain the indecomposable representations of the Lie algebra of the homogeneous Galilei group starting with finite-dimensional representations of the Lorentz algebra $so(1,3)$ and applying the contraction procedure. Special features of this approach are summarized in the following items.

- To obtain *indecomposable* representations of algebra $hg(1,3)$ found in section 4, we were supposed to use *completely reducible* representations of the Lorentz algebra.
- It is possible to obtain different non-equivalent realizations of $hg(1,3)$ starting with a given representation of $so(1,3)$. For example, both realizations $D(2,1,0)$ and $D(2,1,1)$ can be obtained via contractions of the representation $D\left(\frac{1}{2},\frac{1}{2}\right)\oplus D(1,0)$.
- It is possible to obtain a given representation of $hg(1,3)$ via contraction of different representations of $so(1,3)$. For example, the above-mentioned representation $D\left(\frac{1}{2},\frac{1}{2}\right)\oplus D(1,0)$ can be replaced by $D\left(\frac{1}{2},\frac{1}{2}\right)\oplus D(0,1)$, or, more generally, any representation $D(m,n)$ of $so(1,3)$ can be replaced by $D(n,m)$.

7. Galilean linear spin-1/2 wave equation with Pauli anomalous interaction

The complete list of indecomposable spinor and vector representations of algebra $hg(1,3)$ found in sections 3 and 4 can be used to construct Galilei invariant models for spinor and vector fields both linear and nonlinear ones. In particular, these representations can be applied to derive systems of equations invariant with respect to the Galilei group.

In this section, we apply them to describe possible Pauli-type interactions for spinor field invariant with respect to the Galilei group.

7.1. Reduction approach

Consider the Dirac equation describing a fermionic field $\psi(x)$ coupled to a gauge field A_μ and a Pauli anomalous term:

$$(\gamma_\mu\pi^\mu + k[\gamma^\mu,\gamma^\nu]F_{\mu\nu} - \lambda)\psi(x) = 0, \tag{46}$$

where

$$\pi_\mu \equiv p_\mu - qA_\mu$$

and the second term is the Pauli anomalous term.

A natural way to construct a Galilean analogue of equation (46) is to generalize it to the case of (4, 1)-dimensional space and then make the reduction discussed in section 1. However, in this case it is desirable to present a clear physical interpretation for all values obtained by the reduction. In addition, in this way we cannot obtain the most general Pauli interaction term. Thus, we consider two possibilities: the reduction of equation defined in (4, 1)-dimensional space and a direct search for the Pauli term invariant under the Galilei group.

In order to perform the reduction, it is sufficient to change in (46) the Dirac matrices to their Galilean analogues, i.e., to change $\gamma_\mu \to \beta_\mu$ where β_μ are defined by relations (36). In particular, they may be chosen as [25]

$$\beta^0 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}, \qquad \beta^a = \begin{pmatrix} \sigma^a & 0 \\ 0 & -\sigma^a \end{pmatrix}, \qquad \beta^4 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}.$$

If we substitute these matrices into equation (46) for $\psi = \binom{\phi}{\chi}$ (where ϕ and χ are two-component spinors) and use the notations (34), we find

$$\begin{aligned} &(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - \lambda + 4\mathrm{i}k\boldsymbol{\sigma}\cdot\mathbf{W} + 4kP]\varphi + \sqrt{2}(\pi_4 + 4k\boldsymbol{\sigma}\cdot\mathbf{R})\chi = 0, \\ &\sqrt{2}(\pi_0 - 4k\boldsymbol{\sigma}\cdot\mathbf{N})\varphi + (\boldsymbol{\sigma}\cdot\boldsymbol{\pi} + \lambda - 4\mathrm{i}k\boldsymbol{\sigma}\cdot\mathbf{W} + 4kP)\chi = 0, \end{aligned} \tag{47}$$

where $\pi_4 = m - qA_4$. Solving the first of equations (47) for χ and substituting the solution into the second line, we come to the Schrödinger equation with spin-dependent potential which is not discussed here.

In the magnetic limit, defined as in [18],

$$(\mathbf{A}_m, \phi_m) \hookrightarrow A_m = (A_0, \mathbf{A}_m, A_4) = (-\phi_m, \mathbf{A}_m, 0)\,, \tag{48}$$

we find

$$\mathbf{B}_m = \mathbf{W} = \nabla \times \mathbf{A}_m,$$
$$\mathbf{E}_m = \mathbf{N} = -\nabla\phi_m - \partial_t \mathbf{A}_m.$$

Note that $\mathbf{R} = \mathbf{0}$ and $P = 0$, and Galilei transformations for $\mathbf{A_m}$, $\phi_\mathbf{m}$, $\mathbf{B_m}$ and $\mathbf{E}_m$ have the form

$$\begin{aligned} &\mathbf{A}_m \to \mathbf{A}_m, && \phi_m \to \phi_m - \mathbf{v}\cdot\mathbf{A}_m, \\ &\mathbf{B}_m \to \mathbf{B}_m, && \mathbf{E}_m \to \mathbf{E}_m + \mathbf{v}\times\mathbf{B}_m. \end{aligned} \tag{49}$$

With these definitions, equation (47) becomes

$$(\boldsymbol{\sigma}\cdot(\boldsymbol{\pi} - q\mathbf{A}_m) - \lambda + 4\mathrm{i}k\boldsymbol{\sigma}\cdot\mathbf{B}_m)\varphi + \sqrt{2}m\chi = 0,$$
$$\sqrt{2}(p_0 + q\phi_m - 4k\boldsymbol{\sigma}\cdot\mathbf{E}_m)\varphi + [\boldsymbol{\sigma}\cdot(\boldsymbol{\pi} - q\mathbf{A}_m) + \lambda - 4\mathrm{i}k\boldsymbol{\sigma}\cdot\mathbf{B}_m]\chi = 0.$$

This may be rewritten as

$$[\beta_\mu\pi^\mu - \lambda + 2k(\mathbf{S}\cdot\mathbf{B}_m + \boldsymbol{\eta}\cdot\mathbf{E}_m)]\psi = 0, \tag{50}$$

where $\mathbf{S}$ and $\boldsymbol{\eta}$ are matrix vectors whose components are given in equation (37).

Thus, we obtain in a rather elegant way equation (50) which is manifestly Galilei invariant and describes both the minimal and anomalous couplings of a particle of spin 1/2 with an external Galilean electromagnetic field of magnetic type. In the case $k = 0$, i.e., when only the minimal interaction is present, this equation is equivalent to the Lévy-Leblond equation. However, it presents only one of the many possibilities of introducing anomalous coupling into the Lévy-Leblond equation. A more general approach is presented in the following subsection.

7.2. Direct approach

Anomalous Pauli interaction is represented in equation (46) by the term $k[\gamma_\mu, \gamma_\nu]F^{\mu\nu}$, which is linear with respect to the electromagnetic field strength $F^{\mu\nu}$. It is possible to show that the requirement of relativistic invariance defines this term in a unique way (up to a value of the coupling constant k).

In contrast to the above, in the Galilei invariant approach there are more possibilities of introducing the anomalous interaction. In this subsection, we use our knowledge of Galilean vector field (refer to section 4) to find the most general form of anomalous interaction for Galilean spinors.

First, we recall that there exist two types of massless Galilean 4-vector fields, i.e., Ψ_4 and $\tilde{\Psi}_4$, whose transformation properties are defined by equations (23) and (24) with $m = 0$, and a 5-vector Ψ_5 which transforms as given in (32). In other words, we have three types of potentials of external vector field which can be used to introduce the *minimal* interaction into the Lévy-Leblond equation, and they are potentials which generate field strengths involved into anomalous interaction terms whose examples are present in the previous subsection.

We note that in addition it is possible to introduce minimal and anomalous interactions with fields whose potentials are 3-vectors or scalars. Moreover, the Galilei invariance condition admits some constrains for the potentials which generate additional possible anomalous interactions.

Let us start with the vector $\tilde{\Psi}_4$ which corresponds to the magnetic limit field considered in the previous section and find the most general Galilean scalar matrix F linear in $\mathbf{B}_m$ and $\mathbf{E}_m$. Expanding F via the complete set of matrices β_μ (36) and $S_{\mu\nu}$ (38) with well-defined transformation properties and using (49), we easily find that $F = 2k(\mathbf{S}\cdot\mathbf{B}_m + \boldsymbol{\eta}\cdot\mathbf{E}_m) + g\boldsymbol{\eta}\cdot\mathbf{B}_m$ where k and g are arbitrary parameters. In other words, we come to the following Galilei invariant equation for spinor field with the minimal and anomalous interaction:

$$[\beta_\mu\pi^\mu - \lambda + 2k\,(\mathbf{S}\cdot\mathbf{B}_m + \boldsymbol{\eta}\cdot\mathbf{E}_m) + g\boldsymbol{\eta}\cdot\mathbf{B}_m]\psi = 0. \tag{51}$$

In contrast to (50), equation (51) includes two coupling constants, k and g.

The other Galilean limit, named 'electric limit' [6] for the electromagnetic field, corresponds to the gauge fields of type Ψ_4, i.e., the related vector potential

$$(\mathbf{A}_e, \phi_e) \hookrightarrow A_e = (A_0, \mathbf{A}, A_4) = (0, \mathbf{A}_e, \phi_e)\,, \tag{52}$$

and field strengths $\mathbf{B}_e = \nabla\times\mathbf{A}_e$, $\mathbf{E}_e = -\nabla\phi_e$, $\Phi = \partial_0\phi$ transform as

$$\begin{array}{lll} \mathbf{A}_e \to \mathbf{A}_e + \mathbf{v}\phi_e, & \phi_e \to \phi_e & \\ \mathbf{E}_e \to \mathbf{E}_e, & \mathbf{B}_e \to \mathbf{B}_e - \mathbf{v}\times\mathbf{E}_e, & \Phi \to \Phi - \mathbf{v}\cdot\mathbf{E_e}. \end{array} \tag{53}$$

Searching for the related Galilean Dirac equation with a general Pauli interaction term and using the fact that the vectors $(\mathbf{B}_e, -\mathbf{E}_e)$ have the same transformation properties as $(\mathbf{E}_m, \mathbf{B}_m)$, we conclude that to achieve our goal it is sufficient to change $\mathbf{E}_m \to \mathbf{B}_e$ and $\mathbf{B}_m \to -\mathbf{E}_e$ in (51) and add the additional invariant term $\gamma_0\Phi + \boldsymbol{\gamma}\cdot\mathbf{E}_e$. In addition, to keep the Galilei invariance we should introduce the minimal interaction in the following manner:

$$\pi_0 = \mathrm{i}\partial_0, \qquad \pi_a = -\mathrm{i}\partial_a - q(A_e)_a, \qquad \pi_4 = m + q\phi_e.$$

As a result, we obtain

$$(\beta_\mu\pi^\mu - \lambda + 2k\,(\boldsymbol{\eta}\cdot\mathbf{B}_e - \mathbf{S}\cdot\mathbf{E}_e) + g\boldsymbol{\eta}\cdot\mathbf{E}_e + r(\beta_0\phi_e + \boldsymbol{\gamma}\cdot\mathbf{E}_e))\psi = 0. \tag{54}$$

Thus, there exist at least two ways to describe anomalous interaction in the Galilei invariant approach, presented by equations (51) and (54). Equation (51) includes two coupling constants while in (54) the number of such constants is equal to 3.

Note that transformation laws (49) enable us to impose in (51) the Galilei invariant condition $\mathbf{A}_m = 0$ on the vector potential. Thus, there exist one more Galilei invariant equation with anomalous interaction, namely

$$[\beta_0(p_0 + q\phi) - \boldsymbol{\gamma}\cdot\mathbf{p} - \beta_4 m + 2k\boldsymbol{\eta}\cdot\nabla\phi]\,\psi = 0. \tag{55}$$

Analogously, starting with (54) taking into account that transformations (53) are compatible with the condition $\phi_e = 0$ we come to one more Galilei invariant equation with Pauli interaction term, namely

$$(\beta_0\partial_0 - \boldsymbol{\gamma}\cdot\boldsymbol{\pi} - \beta_4 m - \lambda + g\boldsymbol{\eta}\cdot\mathbf{B})\,\psi = 0, \tag{56}$$

where $\boldsymbol{\pi} = -\mathrm{i}\nabla - q\mathbf{A}$, $\mathbf{B} = \nabla\times\mathbf{A}$.

Gauge invariant fields $\mathbf{E} = -\nabla\phi$ and $\mathbf{B} = \nabla\times\mathbf{A}$ present in equations (55) and (56) can be interpreted as external electric and magnetic fields.

Consider also the case where the potential of the external field forms a 5-vector, and related field strengths are given by formulae (33)–(35). We will not discuss possible physical interpretation of the corresponding ten-component vector field but mention that there exists a formal possibility of introducing the anomalous interaction with it by generalizing the Galilean Dirac equation to the following one:

$$(\beta^\mu\pi_\mu + k[\beta^\mu, \beta^\nu]F_{\mu\nu} + g\eta_a R_a + r(S_a R_a + \eta_a U_a) - \lambda)\psi(x) = 0, \tag{57}$$

with three coupling constants k, g, r additional to q.

Like equation (46) with γ-matrices (47), equation (57) is Galilei invariant but is quite more general.

Finally, let the external field is defined by a 5-vector potential satisfying the Galilei invariant constraint

$$\partial_a A_4 = 0, \tag{58}$$

then $F_{4a} \equiv 0$ and tensor $F_{\mu\nu}$ (34) is reduced to the 7-vector $\tilde{\Psi}_7 = (\mathbf{K}, \mathbf{R}, A)$ whose components are

$$\mathbf{K} = \nabla A_0 - \partial_0 \mathbf{A}, \qquad \mathbf{R} = \nabla \times \mathbf{A}, \qquad A = \partial_0 A_4.$$

To find the corresponding invariants linear in $F_{\mu\nu}$, it is necessary to construct invariant scalar products of vector functions (58) and matrices (39) belonging to $\tilde{\Psi}_7$. As a result, we come to the following equation:

$$(\beta^\mu \pi_\mu + g\eta_a R_a + r(S_a R_a + \eta_a K_a + S^{04} A) + \nu \beta_0 A - \lambda)\psi(x) = 0, \tag{59}$$

where g, r and ν are coupling constants. For $A = 0$, equation (59) is reduced to equation (51) which describes anomalous interaction of the Galilei particle of spin 1/2 with an external field of magnetic type.

Formulae (51), (54)–(57) and (59) present all non-equivalent Galilean invariant equations for spinor field, describing minimal and anomalous interactions with external gauge fields.

We see that the direct search for the Galilei invariant Pauli interaction makes it possible to find a more general coupling than the reduction method.

7.3. Galilean system with spin–orbit coupling

In this section, we consider one of the described systems and discuss its physical content.

Let us start with equation (55) which describes interaction of the Galilean spinor particle with an external electric field. Choosing β-matrices in the form (47) and denoting $k = -\frac{q\hat{k}}{4m}$, we write this equation componentwise

$$(\boldsymbol{\sigma}\cdot\mathbf{p} - \lambda)\varphi + \sqrt{2}m\chi = 0,$$
$$\sqrt{2}\left(p_0 + q\phi + \frac{q\hat{k}}{m}\boldsymbol{\sigma}\cdot\mathbf{E}\right)\varphi + (\boldsymbol{\sigma}\cdot\mathbf{p} + \lambda)\chi = 0,$$

where $\mathbf{E} = -\nabla\phi$.

Solving the first equation for χ and substituting the result into the second equation, we obtain

$$L\varphi \equiv \left(p_0 - \frac{p^2 + \lambda^2}{2m} + q\phi - \frac{q\hat{k}}{m}\boldsymbol{\sigma}\cdot\mathbf{E}\right)\varphi = 0. \tag{60}$$

In other words, we come to the Galilei invariant Schrödinger equation with a matrix potential.

To analyse the physical content of equation (60), we transform it to a more familiar form using the operator $U = \exp\left(-\frac{\mathrm{i}\hat{k}}{m}\boldsymbol{\sigma}\cdot\mathbf{p}\right)$. Applying this operator to φ and L, we obtain the equation $L'\varphi' = 0$ where $\varphi' = U\varphi$,

$$L' = ULU^{-1} = p_0 - \frac{p^2}{2m} + q\phi - \frac{\lambda^2}{2m} - \frac{q\hat{k}^2}{2m^2}(\boldsymbol{\sigma}\cdot(\mathbf{p}\times\mathbf{E} - \mathbf{E}\times\mathbf{p}) - \mathrm{div}\mathbf{E}) + \cdots,$$

where the dots denote the terms of order $o\left(\frac{1}{m^3}\right)$.

All terms in the operator L' have exact physical meaning. In particular, the last two terms describe spin–orbit and Darwin couplings of a Galilean particle with an external field.

8. Discussion

The relativity principle is one of the corner stones of modern physics. Moreover, it should be applied not only while considering phenomena characterized by circumlight velocities but also in the cases where the velocities are small in comparison with the velocity of light. In the latest case, any well-formulated physical theory should not be simply 'non-relativistic', but should satisfy the Galilei relativity principle. In other words, the group of motion of the special relativity (i.e., the Lorentz group) should be replaced by the Galilei group.

It appears that a consistent use of the Galilei group and its representations in many aspects is much more complicated than in the case of Poincaré group. In particular, description of finite-dimensional representations of the homogeneous Galilei group is a wild algebraic problem while such representations for the Lorentz group have been found long time ago.

The main goal of the present paper was to present the complete classification of indecomposable Galilean fields which transform as vectors or are scalars with respect to the rotation transformations. In other words, we present the classification of indecomposable finite-dimensional representations of the homogeneous Galilei group which being reduced to the rotation subgroup correspond to spins $s \leqslant 1$. In contrast to the fields with spin $s > 1$, the Galilean vectors' field can be described completely. The results of this classification are presented in section 3.

In contrast to the Lorentz vectors, the number of non-equivalent Galilean vectors appears to be rather extended. Namely, for the Lorentz group there exist the following indecomposable vector fields: 4-vector, bi-vector (i.e., antisymmetric tensor of second rank) and two 3-vectors which are nothing but self-dual and anti-self-dual parts of the antisymmetric tensor. In the case of Galilei group, it is possible to indicate nine indecomposable vector fields.

We use our knowledge of vector representations of the Galilei group to describe all possible Pauli interactions for spinor fields, compatible with the Galilei invariance. The number of such interactions appears to be rather extended in contrast to the relativistic approach where this interaction is unique up to the coupling constant. We show that there exist such Galilei invariant systems which describe spin–orbit and Darwin couplings which are traditionally treated as pure relativistic effects. A collection of other Galilean systems with spin–orbit coupling can be found in [14].

It is generally accepted to think that the Galilei group and its representations can easily be obtained starting with representations of the Lorentz group and making the Inönü–Wigner contraction [7]. We had shown that this procedure is not too straightforward in as much as starting with indecomposable representations of the Lorentz group we can obtain only a part of the corresponding representations of the homogeneous Galilei group. On the other hand, it is possible to contract completely reducible representations of the Lorentz group to indecomposable representations of the homogeneous Galilei group.

The complete list of vector–scalar and spinor indecomposable representations presented in sections 3 and 4 opens a way to construct Galilei invariant models for scalar, spinor and vector fields. For this purpose, it is important to study tensor products of the described representations. This work is in progress; some results will be announced in [26]. In the following paper, we also plan to discuss Galilei invariant equations for spinor and vector fields and study the relationship between relativistic and Galilean approaches using the contraction procedures described above.

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Appendix. Solution of the matrix problem (16), (17)

Here we find all indecomposable matrices A, B and C of dimensions $n \times n$, $n \times m$ and $m \times m$, respectively, which are defined up to equivalence relations (18).

Multiplying equation (16) by A and using equation (17), we obtain the condition $A^3 = 0$; hence, A is a nilpotent matrix whose nilpotency index N satisfies the condition $N \leqslant 3$. This implies that BC is a nilpotent matrix with index of nilpotency equal to 2. Thus, A can be represented as a direct sum of the Jordan cells of dimension 3, 2 and 0 matrices. We shall prove (see lemma 3) that in order to obtain indecomposable representations of algebra defined by relations (4)–(6) it is necessary to restrict ourselves to the case of indecomposable matrices A, and so there are three possibilities

$$A = A^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A = A^{(2)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A = A^{(3)} = 0. \tag{A.1}$$

First, we suppose *a priori* that matrices A are indecomposable and find the corresponding non-equivalent sets of matrices A, B, C satisfying (16), (17). To do this, we consider consequently all matrices A enumerated in (A.1).

Let $A = A^{(1)}$, then $n = 3$ and so B and C are matrices of dimension $3 \times m$ and $m \times 3$. It follows from (17) and (16) that

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ b_1 & b_2 & \cdots & b_m \end{pmatrix}, \qquad C^T = \begin{pmatrix} c_1 & c_2 & \cdots & c_m \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{A.2}$$

and

$$b_1 c_1 + b_2 c_2 + \cdots + b_m c_m = -1,$$

where C^T is a matrix transposed to C.

Up to equivalence transformations (18), we can choose

$$b_m = 1, \qquad c_1 = -1, \qquad b_2 = b_3 = \cdots = b_m = 0, \qquad c_2 = c_3 = \cdots = c_m = 0. \tag{A.3}$$

Substituting (A.2), (A.3) into (14), we come to matrices η_a which are direct sums of 10×10 indecomposable matrices and $(m-1) \times (m-1)$ zero matrices. The related set of matrices $\{S_a, \eta_a\}$ with S_a given in (12) is indecomposable iff $m = 1$.

Certainly, for $m > 1$ the obtained realization for matrices A, B and C is completely reducible too, since we can treat as A a direct sum of $A^{(1)}$ and the zero matrix of an appropriate dimension.

If $A = A^{(2)}$, then B and C are matrices of dimensions $2 \times m$ and $m \times 2$, respectively. Moreover, in accordance with (17) and (16)

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ b_1 & b_2 & \cdots & b_m \end{pmatrix}, \qquad C^T = \begin{pmatrix} c_1 & c_2 & \cdots & c_m \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{A.4}$$

and

$$b_1c_1 + b_2c_2 + \cdots + b_mc_m = 0. \tag{A.5}$$

Up to equivalence transformations (18), we can specify four non-equivalent solutions for equations (A.5):

$$b_1 = b_2 = b_3 = \cdots = b_m = 0, \qquad c_1 = c_2 = \cdots = c_m = 0, \tag{A.6}$$

$$\begin{aligned} &b_1 = 1, \qquad b_2 = b_3 = \cdots = b_m = 0, \qquad c_1 = c_2 = \cdots = c_m = 0,\\ &b_1 = b_2 = b_3 = \cdots = b_m = 0, \qquad c_1 = 1, \qquad c_2 = \cdots = c_m = 0, \end{aligned} \tag{A.7}$$

$$b_1 = 1, \qquad b_2 = b_3 = \cdots = b_m = 0, \qquad c_1 = 1, \qquad c_2 = \cdots = c_m = 0. \tag{A.8}$$

Repeating the reasoning which follows equation (A.3), we conclude that to obtain indecomposable sets of the related matrices η_a and S_a it is necessary to set $m = 0$, $m = 1$ and $m = 2$ for solutions (A.6), (A.7) and (A.8), respectively.

For $A = A^{(1)}$, we have

$$B = (b_1\ b_2\ \cdots\ b_m), \qquad C^T = (c_1\ c_2\ \cdots\ c_m),$$

where parameters $b_1, b_2, \ldots, b_m$ and $c_1, c_2, \ldots, b_m$ should satisfy relations (A.5). Thus, we again have four solutions enumerated in equations (A.6)–(A.8) with the same restrictions for values of m.

Let us summarize the results of our analysis as the following assertion.

Lemma 2. *Let A be an indecomposable matrix of dimension $n \times n$. Then up to equivalence transformations (18) there exist eight indecomposable sets of matrices $\{A, B, C\}$ (dimension of B is $n \times m$ and dimension of C is $m \times n$) satisfying relations (16), (17). These sets can be enumerated by triplets of numbers $(n, m, \lambda = \text{Rank}\, B)$, whose possible values are given in (19), (20) and have the following form:*

1. $(n, m, \lambda) = (0, 0, 0)$, *$A$, B and C do not exist.*
2. $(n, m, \lambda) = (1, 0, 0)$, $A = 0$, *B and C do not exist.*
3. $(n, m, \lambda) = (1, 1, 0)$, $A = 0$, $B = 0$, $C = 1$.
4. $(n, m, \lambda) = (1, 1, 1)$, $A = 0$, $B = 1$, $C = 0$.
5. $(n, m, \lambda) = (1, 2, 1)$, $A = 0$, $B = (10)$, $C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
6. $(n, m, \lambda) = (2, 0, 0)$, $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, *$B$ and C do not exist.*
7. $(n, m, \lambda) = (2, 1, 0)$, $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $C = (10)$.
8. $(n, m, \lambda) = (2, 1, 1)$, $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C = (00)$.
9. $(n, m, \lambda) = (2, 2, 1)$, $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
10. $(n, m, \lambda) = (3, 1, 1)$, $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $C = (100)$.

To complete the solution of equations (16), (17), we shall prove the following proposition.

Lemma 3. *Let matrices A, B and C satisfy relations (16), (17) and A is decomposable. Then the set of matrices $\{A, B, C\}$ is decomposable too.*

Proof. Since A is a nilpotent matrix satisfying $A^3 = 0$, it can be reduced to a direct sum of Jordan cells presented in (A.1) (any of the cells can be present with some multiplicity $M = 0, 1, 2, \ldots$).

Let this direct sum include at least one matrix of type $A^{(1)}$, then we can represent the related matrices A, B and C in the following form:

$$A = \begin{pmatrix} A^{(1)} & \mathbf{0}_{3\times(n-3)} \\ \mathbf{0}_{(n-3)\times 3} & \tilde{A} \end{pmatrix}, \qquad B = \begin{pmatrix} B^{(1)} \\ \tilde{B} \end{pmatrix}, \qquad C = (C^{(1)} \tilde{C}), \tag{A.9}$$

where $B^{(1)}, \tilde{B}, C^{(1)}$ and $\tilde{C}$ are matrices of dimensions $3 \times m, (n-3) \times m, m \times 3$ and $m \times (n-3)$, respectively.

Substituting (A.9) into (16) and (17), we come to the following relations:

$$A^{(1)2} + B^{(1)}C^{(1)} = 0, \; A^{(1)}B^{(1)} = 0, \; C^{(1)}A^{(1)} = 0, \tag{A.10}$$

$$B^{(1)}\tilde{C} = 0, \qquad \tilde{B}C^{(1)} = 0, \tag{A.11}$$

$$\tilde{A}^2 + \tilde{B}\tilde{C} = 0, \qquad \tilde{A}\tilde{B} = 0, \qquad \tilde{C}\tilde{A} = 0. \tag{A.12}$$

Up to equivalence transformations (18), the general solutions of equations (A.10) are given by relations (A.2), (A.3), which yield the following consequences of equations (A.11) for the general form of matrices $\tilde{B}$ and $\tilde{C}$:

$$\tilde{B} = (\mathbf{0}_{(n-3)\times 1} \hat{B}_{(n-3)\times(m-1)}), \qquad \tilde{C} = \begin{pmatrix} \mathbf{0}_{1\times(n-3)} \\ \hat{C}_{(n-1)\times(m-3)} \end{pmatrix}, \tag{A.13}$$

where $\hat{B}_{(n-3)\times(m-1)}$ and $\hat{C}_{(n-1)\times(m-3)}$ are matrices whose dimension is indicated in the sub-indices. In the following, these sub-indices will be omitted.

Substituting (A.13) into (A.12), we obtain the standard relations for $\hat{B}$, $\hat{C}$ and $\tilde{A}$:

$$\tilde{A}^2 + \hat{B}\hat{C} = 0, \qquad \tilde{A}\hat{B} = 0, \qquad \hat{C}\tilde{A} = 0. \tag{A.14}$$

In accordance with (A.9), (A.10), (A.13), (A.14), the matrices A, B and C are nothing but direct sums of matrices satisfying relations (16), (17):

$$A = A^{(1)} \oplus \tilde{A}, \qquad B = B^{(1)} \oplus \tilde{B}, \qquad C = C^{(1)} \oplus \tilde{C},$$

so the set of matrices (A.9) is decomposable.

In an analogous way (but using equivalence transformations (18) also), we prove the complete reducibility of matrices $\{A, B, C\}$ if A is decomposable and $A^2 = 0$. To do this, it is sufficient to use the decomposition of type (A.9) where $A^{(1)} \to A^{(2)}$, $A^{(2)}$ is the 2×2 matrix given in (A.1) and $B^{(1)}, C^{(1)}$ are matrices defined by equations (A.4) and (A.6). Finally, for the case where A is the zero matrix of dimension $n > 1$, the proof of complete reducibility of the related matrices B and C reduces to direct use of equivalence transformations (18). □

We see that *a priori* requirement that matrix A is indecomposable which we use in lemma 2 does not lead to loss of generality and so the list of matrices presented there is complete. Substituting these matrices into formula (14) and considering the related spin matrices (12), we obtain the list of basis elements of the algebra $hg(1, 3)$ which is given in table 1.

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